Pergamon

Computers Math. Applic. Vol. 32, No. 12, pp. 29–39, 1996

0898-1221/96 $15.00 + 0.00

PII: S0898-1221(96)00205-2

Oscillation of Nonlinear Partial Difference Equations with Delays

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(Received July 1996; accepted August 1996)

Abstract—In this paper, we first derive a discrete Gaussian formula and then apply the formula to two classes of nonlinear (parabolic and hyperbolic) partial difference equations with delays to obtain sufficient conditions under which every solution of the two classes of nonlinear partial difference equations with delays is oscillatory.

Keywords—Oscillation, Parabolic partial difference equations with delays, Hyperbolic partial difference equations with delays, Neutral type, Discrete Gaussian formula.

1. INTRODUCTION

Recently, the oscillation of partial differential equations with delays has been extensively investigated (see, for instance, [1–15], etc.). The stability and asymptotics for partial differential equations with delays have also been widely approached (see, for example, [16–18] and the authors' [19], etc.). However, the works on partial difference equations with delays are very few in the literature (see [20,21] and the authors' [19,22]), in particular, on the oscillation of partial difference equations with delays (see [23]).

It is well known that the behavior of a differential equation and its discrete analogue can be quite different. For example, every solution of the Logistic equation

$$x'(t) = rx(t)\left[1 - \frac{x(t)}{K}\right]$$

is monotonic. But its discrete analogue

$$x_{n+1} = m\,x_n\,(1 - x_n)$$

has a chaotic solution when $m = 4$ (see [24]). In addition, the difference on the oscillation of delay differential equations and its discrete analogues also exists (see [25]).

Our aim in this paper is to obtain some sufficient conditions under which every solution of the two classes of nonlinear partial difference equations with delays, which will be shown in Sections 3 and 4, respectively, is oscillatory.

The Project supported by NSFC.

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2. DISCRETE GAUSSIAN FORMULA

Consider a sequence $\{u_{m,n}\} := \{u_{m_1,\dots,m_l,n}\}$ which is defined on $\Omega \times Z^+(n_0)$, where $Z^+(n_0) := \{n_0, n_0+1, \dots\}, \Omega := \{p_1^{(1)}, \dots, p_{M_1}^{(1)}\} \times \cdots \times \{p_1^{(l)}, \dots, p_{M_l}^{(l)}\}$, and every $p_i^{(j)} \in Z := \{\dots, -1, 0, 1, \dots\}$.

Now we give some definitions for deriving the discrete Gaussian formula.

DEFINITION 2.1. *m is said to be an interior point of Ω, if $m+1 := \{m_1+1, m_2, \dots, m_l\} \cup \cdots \cup \{m_1, \dots, m_{l-1}, m_l+1\}$ and $m-1 := \{m_1-1, m_2, \dots, m_l\} \cup \cdots \cup \{m_1, \dots, m_{l-1}, m_l-1\}$ are all in Ω; Ω^0, which is composed of all interior points, is said to be an interior of Ω.*

DEFINITION 2.2. *m is said to be a convex boundary point of Ω, if $m \in \Omega$ and at least l points of $m \pm 1$ are in Ω; m is said to be a concave boundary point, if $m, m \pm 1 \in \Omega$ but just one of the points $\{m_1 \pm 1, \dots, m_l \pm 1\}$ is not in Ω, where $\{m_1 \pm 1, \dots, m_l \pm 1\} := \{m_1+1, \dots, m_l+1\} \cup \{m_1-1, m_2+1, \dots, m_l+1\} \cup \cdots \cup \{m_1-1, \dots, m_l-1\}$; $\partial\Omega$, which is composed of all (convex and concave) boundary points, is said to be a boundary of Ω.*

REMARK 2.1. If Ω is a rectangular solid net (one can see any book on the computation of partial differential equations for the definition), then $\partial\Omega$ is only composed of all convex boundary points.

DEFINITION 2.3. *Ω is said to be convex, if $\partial\Omega$ is only composed of all convex points.*

REMARK 2.2. If Ω is a rectangular solid net, then Ω is convex.

DEFINITION 2.4. *m is said to be an exterior point, if it is neither an interior point nor a boundary point.*

DEFINITION 2.5. *m is said to be an allowable point, if at least two points of $m \pm 1$ are in Ω.*

DEFINITION 2.6. *Ω is said to be a connected net, if Ω is only composed of all allowable points.*

REMARK 2.3. If Ω is a rectangular solid net, then it is a convex connected solid net.

We only consider in this paper that Ω is a convex connected solid net.

DEFINITION 2.7. *If $m \in \partial\Omega$ is a convex boundary point of Ω, we define that the normal difference at $(m,n) \in \partial\Omega \times Z^+(n_0)$ is $\Delta_N u_{m-1,n} := \sum_{\text{all } m\pm 1 \notin \Omega} (\Delta_1 u_{m,n} - \Delta_1 u_{m-1,n}) = \sum_{\text{all } m\pm 1 \notin \Omega} \Delta_1^2 u_{m,n}$, where Δ_1 and Δ_1^2 are, respectively, partial difference operators of order one and of order two (see [24] or the authors' [19,22]).*

We write ∇^2 a discrete Laplacian operator, which is defined by $\nabla^2 u_{m-1,n+1} := \sum_{i=1}^{l} \Delta_i^2 u_{m_1,\dots,m_{i-1},m_i-1,m_{i+1},\dots,m_l,n+1}$, where Δ_i^2 is a partial difference operator of order two.

Now we give the discrete Gaussian formula as follows.

THEOREM 2.1. *(Discrete Gaussian Formula) Let Ω be a convex connected solid net. Then we have*

$$\sum_{m\in\Omega} \nabla^2 u_{m-1,n+1} = \sum_{m\in\partial\Omega} \Delta_N u_{m-1,n+1}. \tag{1}$$

PROOF. Because a convex connected solid net can be divided into several rectangular solid nets, therefore, we can only consider the latter case. Without loss of generality, we let $\Omega := \{1, \dots, M_1\} \times \cdots \times \{1, \dots, M_l\}$. In what follows, we give only, for the sake of simplicity, the proof in the case of $l = 2$.

$$\begin{aligned}
\sum_{m\in\Omega} \nabla^2 u_{m-1,n+1} &= \sum_{m\in\Omega} \left(\Delta_1^2 u_{m_1-1,m_2,n+1} + \Delta_2^2 u_{m_1,m_2-1,n+1}\right) \\
&= \sum_{m\in\Omega} (u_{M_1,m_2,n+1} - u_{M_1,m_2,n+1} - u_{1,m_2,n+1} + u_{0,m_2,n+1} + u_{m_1,M_2+1,n+1} \\
&\quad -u_{m_1,M_2,n+1} - u_{m_1,1,n+1} + u_{m_1,0,n+1}) \\
&= \sum_{m\in\Omega} (\Delta_1 u_{m_1,m_2,n+1}|_{m_1=M_1} - \Delta_1 u_{m_1,m_2,n+1}|_{m_1=0} + \Delta_2 u_{m_1,m_2,n+1}|_{m_2=M_2} \\
&\quad - \Delta_2 u_{m_1,m_2,n+1}|_{m_2=0}) \\
&= \sum_{m_2=1}^{M_2} (\Delta_1 u_{m_1,m_2,n+1}|_{m_1=M_1} - \Delta_1 u_{m_1,m_2,n+1}|_{m_1=0}) \\
&\quad + \sum_{m_1=1}^{M_1} (\Delta_2 u_{m_1,m_2,n+1}|_{m_2=M_2} \Delta_2 u_{m_1,m_2,n+1}|_{m_2=0}).
\end{aligned}$$

Noting that the first term in the above is the sum of the normal differences on both left and right boundaries and the second one on both upper and lower boundaries of Ω, we have that the equality (1) holds and complete the proof.

3. PARABOLIC EQUATIONS

We consider in this section the nonlinear parabolic difference equations of neutral type of the form

$$\begin{gathered}
\Delta_2\left(u_{m,n} - \sum_{k\in K} r_{k,n} u_{m,n-\alpha_k}\right) + p_{m,n} u_{m,n} + \sum_{i\in I} p_{m,n}^{(i)} f_i\left(u_{m,n-\beta_i}\right) \\
= q_n \nabla^2 u_{m-1,n+1} + \sum_{j\in J} q_{j,n} \nabla^2 u_{m-1,n+1-\gamma_j}, \\
\text{for } m\in\Omega \quad \text{and} \quad n\in Z^+(n_0),
\end{gathered} \tag{2}$$

where $I := \{1,\dots,I_0\}$, $J := \{1,\dots,J_0\}$, $K := \{1,\dots,K_0\}, \Omega$ is a convex connected net, and $Z^+(n_0) := \{n_0, n_0+1,\dots\}$.

We assume throughout this section that

(H$_1$) $q_n \in Z^+(n_0) \to \mathbf{R}^+$ and $q_{j,n} \in J\times Z^+(n_0) \to \mathbf{R}^+$;

(H$_2$) $p_{m,n} \in \Omega\times Z^+(n_0) \to \mathbf{R}^+$, $p_{m,n}^{(i)} \in I\times\Omega\times Z^+(n_0) \to \mathbf{R}^+$, $p_n := \min_{m\in\Omega}\{p_{m,n}\}$, and $p_{i,n} := \min_{m\in\Omega}\{p_{m,n}^{(i)}\}$, for $i\in I$ and $n\in Z^+(n_0)$;

(H$_3$) $\alpha_k \in K \to Z^+(1)$, $\beta_i \in I \to Z^+(1)$, and $\gamma_j \in J \to Z^+(1)$;

(H$_4$) $f_i \in C(\mathbf{R}\to\mathbf{R})$ are convex, increasing on $\mathbf{R}^+\setminus\{0\}, uf_i(u) > 0$ for $u\neq 0$ and $i\in I$, and $f(0)=0$;

(H$_5$) $r_{k,n} \in K\times Z^+(n_0) \to \mathbf{R}^+$ and $\sum_{k\in K} r_{k,n} \le 1$.

Consider the initial boundary value problem (IBVP)(2) with the homogeneous Rodin boundary condition (RBC)

$$\Delta_N u_{m-1,n} + g_{m,n} u_{m,n} = 0, \qquad \text{on } \partial\Omega\times Z^+(n_0), \tag{3}$$

and the initial condition (IC)

$$u_{m,s} = \mu_{m,s}, \qquad \text{for } n_0-\tau \le s \le n_0, \tag{4}$$

where $\tau = \max\{\alpha_k, \beta_i, \gamma_j : k\in K, i\in I \text{ and } j\in J\}$ and $g_{m,n} \in \partial\Omega\times Z^+(n_0) \to \mathbf{R}^+$.

By a solution of IBVP(2)–(4), we mean a sequence $\{u_{m,n}\}$ which satisfies equation (2) for $(m,n)\in\Omega\times Z^+(n_0)$, satisfies RBC(3) for $(m,n)\in\partial\Omega\times Z^+(n_0)$, and satisfies IC(4) for $(m,n)\in\Omega\times\{n_0-\tau,\dots,n_0\}$. For the (unique) existence of solutions, one is referred to [22].

Our objective in this section is to present sufficient conditions which imply that every solution $\{u_{m,n}\}$ of IBVP(2)–(4) is oscillatory in $\Omega \times Z^+(n_0)$, in the sense that there exists not an $n' \in Z^+(n_0)$ such that $u_{m,n} > 0$ or $u_{m,n} < 0$ for $n \in Z^+(n')$.

THEOREM 3.1. *Let hypotheses (H₁)–(H₅) hold. Suppose that there exist two constants $B, C > 0$, and an $i_0 \in I$ such that $f_{i_0}(u)/u \geq C$ for $u \neq 0$, and $p_n, p_{i_0,n} \geq B$ for $n \in Z^+(n_0)$. If*

$$\limsup_{n\to\infty} \sum_{s=n-\beta_{i_0}}^{n} p_{i_0,s} > \frac{1}{C}, \tag{5}$$

then every solution $\{u_{m,n}\}$ of IBVP(2)–(4) is oscillatory in $\Omega \times Z^+(n_0)$.

PROOF. Suppose that it is not the case and $\{u_{m,n}\}$ is a nonoscillatory solution. Without loss of generality, we may assume that there exists an $n_1 \in Z^+(n_0)$ such that $u_{m,n} > 0$ for $n \in Z^+(n_1)$. Hence, $u_{m,n-\alpha_k}$, $u_{m,n-\beta_i}$, and $u_{m,n-\gamma_j} > 0$ for $n \in Z^+(n_1 + \tau) := Z^+(n_2)$.

Summing equation (2) over Ω, we have

$$\begin{gathered}\Delta_2\left(\sum_{m\in\Omega} u_{m,n} - \sum_{k\in K} r_{k,n} \sum_{m\in\Omega} u_{m,n-\alpha_k}\right) + \sum_{m\in\Omega} p_{m,n}u_{m,n} + \sum_{i\in I}\sum_{m\in\Omega} p^{(i)}_{m,n} f_i\left(u_{m,n-\beta_i}\right) \\ = q_n \sum_{m\in\Omega} \nabla^2 u_{m-1,n+1} + \sum_{j\in J} q_{j,n} \sum_{m\in\Omega} \nabla^2 u_{m-1,n+1-\gamma_j}, \\ \text{for } (m,n) \in \Omega \times Z^+(n_2).\end{gathered} \tag{6}$$

From (H₄), Theorem 2.1 and the Jensen's inequality, it follows that

$$\begin{gathered}\sum_{m\in\Omega} \nabla^2 u_{m-1,n+1} = \sum_{m\in\partial\Omega} \Delta_N u_{m-1,n+1} = -\sum_{m\in\partial\Omega} g_{m,n+1}u_{m,n+1} \leq 0, \\ \text{for } n \in Z^+(n_2),\end{gathered} \tag{7}$$

$$\begin{gathered}\sum_{m\in\Omega} \nabla^2 u_{m-1,n+1-\gamma_j} = \sum_{m\in\partial\Omega} \Delta_N u_{m-1,n+1-\gamma_j} = -\sum_{m\in\partial\Omega} g_{m,n+1-\gamma_j}u_{m,n+1-\gamma_j} \leq 0, \\ \text{for } j \in J \quad\text{and}\quad n \in Z^+(n_2),\end{gathered} \tag{8}$$

$$\sum_{m\in\Omega} p_{m,n}u_{m,n} \geq p_n \sum_{m\in\Omega} u_{m,n} = |\Omega| p_n v_n, \qquad \text{for } n \in Z^+(n_2), \tag{9}$$

where $v_n = 1/|\Omega| \sum_{m\in\Omega} u_{m,n}$ and $|\Omega|$ is the number of points in Ω, and

$$\begin{gathered}\sum_{m\in\Omega} p^{(i)}_{m,n} f_i\left(u_{m,n-\beta_i}\right) \geq p_{i,n} \sum_{m\in\Omega} f_i\left(u_{m,n-\beta_i}\right) \geq p_{i,n} f_i\left(\frac{1}{|\Omega|}\sum_{m\in\Omega} u_{m,n-\beta_i}\right)|\Omega|, \\ \text{for } i \in I \quad\text{and}\quad n \in Z^+\left(n_2\right).\end{gathered} \tag{10}$$

Thus, we obtain by (6)–(10) that

$$\Delta\left(v_n - \sum_{k\in K} r_{k,n} v_{n-\alpha_k}\right) + \sum_{i\in I} p_{i,n} f_i\left(v_{n-\beta_i}\right) \leq 0, \qquad \text{for } n \in Z^+(n_2), \tag{11}$$

where Δ is the ordinary difference operator.

Writing that $w_n = v_n - \sum_{k\in K} r_{k,n} v_{n-\alpha_k}$, we have by (H₅) and (11)

$$\Delta w_n < 0 \quad\text{and}\quad w_n \leq v_n. \tag{12}$$

This follows $\lim_{n\to\infty} w_n = L$. We can prove that $L > -\infty$. In fact, if $L = -\infty$, then v_n is unbounded. Hence, there exists an $n_3 \in Z^+(n_2)$ such that

$$w_{n_3} < 0 \quad \text{and} \quad v_{n_3} = \max_{n_2 \le n \le n_3} v_n. \tag{13}$$

It then follows from (H_5) that

$$w_{n_3} - \sum_{k\in K} r_{k,n} v_{n_3-\alpha_k} \ge v_{n_3}\left(1 - \sum_{k\in K} r_{k,n}\right) \ge 0,$$

which contradicts (13). Hence, $L > -\infty$ and is finite.

Summing (11) over $\{n_3, n_3+1, \dots, n\}$, we obtain

$$\begin{aligned} 0 < B \sum_{s=n_3}^{n} f_{i_0}\left(v_{s-\beta_{i_0}}\right) &\le \sum_{i\in I} \sum_{s=n_3}^{n} p_{i,s} f_i\left(v_{s-\beta_i}\right) \\ &\le -\sum_{s=n_3}^{n} \Delta w_s = w_{n_3} - w_{n+1} \le w_{n_3} - L < \infty. \end{aligned}$$

Therefore $f_{i_0}(v_{n-\beta_{i_0}})$ is summable and $\lim_{n\to\infty} v_n = 0$ by (H_4). It then follows that $\lim_{n\to\infty} w_n = 0$.

From (11) and (12), there exists an $n_4 \in Z^+(n_3)$ such that

$$\Delta w_n + \sum_{i\in I} p_{i,n} f_i\left(w_{n-\beta_i}\right) \le 0, \qquad \text{for } n \in Z^+(n_4). \tag{14}$$

Moreover,

$$\Delta w_n + p_{i_0,n} f_{i_0}\left(w_{n-\beta_{i_0}}\right) \le 0, \qquad \text{for some } i_0 \in I \text{ and } n \in Z^+(n_4). \tag{15}$$

Summing (15) from $n - \beta_{i_0}$ to n, we have

$$w_{n+1} - w_{n-\beta_{i_0}} + \sum_{s=n-\beta_{i_0}}^{n} f_{i_0}\left(w_{s-\beta_{i_0}}\right), \qquad \text{for } n \in Z^+\left(n_4\right).$$

Since $\Delta w_n < 0$ and $f_{i_0}(u)$ is increasing on $\mathbf{R}^+ \setminus \{0\}$, we have

$$w_{n+1} - w_{n-\beta_{i_0}} + f_{i_0}\left(w_{n-\beta_{i_0}}\right) \sum_{s=n-\beta_{i_0}}^{n} p_{i_0,s} \le 0, \qquad \text{for } n \in Z^+\left(n_4\right)$$

and

$$\frac{f_{i_0}\left(w_{n-\beta_{i_0}}\right)}{w_{n-\beta_{i_0}}} \sum_{s=n-\beta_{i_0}}^{n} p_{i_0,s} \le 1 - \frac{w_{n+1}}{w_{n-\beta_{i_0}}} < 1.$$

Hence,

$$C \sum_{s=n-\beta_{i_0}}^{n} p_{i_0,s} < 1$$

and

$$\limsup_{n\to\infty} \sum_{s=n-\beta_{i_0}}^{n} p_{i_0,s} \le \frac{1}{C},$$

which contradicts (5). This completes the proof.

THEOREM 3.2. *Let (H_1)–(H_5) hold. Suppose that there exist $C_i \geq 0$ and a $B > 0$, such that $f_i(u)/u \geq C_i$ for $u \neq 0$ and $p_{i_0,n} \geq B$, for some $i_0 \in I$. If*

$$\limsup_{n\to\infty} \sum_{s=n-\tau}^{n} \sum_{i\in I} C_i p_{i,s} > 1, \tag{16}$$

then every solution of IBVP(2)–(4) is oscillatory in $\Omega \times Z^+(n_0)$.

PROOF. Let $\{u_{m,n}\}$ be a nonoscillatory solution of IBVP(2)–(4). Without loss of generality, we assume that $u_{m,n} > 0$ for some $n_5 \in Z^+(n_4)$. Hence, we have $u_{m,n-\alpha_k}, u_{m,n-\beta_i}$ and $u_{m,n-\gamma_j} > 0$, for $n \in Z^+(n_5+\tau) := Z^+(n_6)$. As in the proof of Theorem 3.1, we know that (11)–(14) hold. Summing (14) from $n-\beta$ to n, we have

$$w_{n+1} - w_{n-\beta} + \sum_{i\in I} \sum_{s=n-\beta}^{n} p_{i,s} f_i\left(w_{s-\beta_i}\right) \leq 0, \qquad \text{for } n \in Z^+\left(n_6\right),$$

where $\beta := \max_{i\in I}\{\beta_i\}$ and n_6 is sufficiently large. From (13), we have

$$w_{n+1} - w_{n-\beta} + \sum_{i\in I} f_i\left(w_{n-\beta}\right) \sum_{s=n-\beta}^{n} p_{i,s} \leq 0, \qquad \text{for } n \in Z^+\left(n_6\right).$$

This follows that

$$\sum_{i\in I} \frac{f_i\left(w_{n-\beta}\right)}{w_{n-\beta}} \sum_{s=n-\beta}^{n} p_{i,s} \leq 1 - \frac{w_{n+1}}{w_{n-\beta}} < 1, \qquad \text{for } n \in Z^+\left(n_6\right)$$

and

$$\sum_{s=n-\beta}^{n} \sum_{i\in I} C_i p_{i,s} \leq 1,$$

which contradicts (16). The proof is thus complete.

COROLLARY 3.1. *Let (H_1)–(H_5) hold. If the difference inequality (11) (respectively, (14)) has no ultimately positive solutions, then every solution $\{u_{m,n}\}$ of IBVP(2)–(4) is oscillatory in $\Omega \times Z^+(n_0)$.*

4. HYPERBOLIC EQUATIONS

We consider in this section the nonlinear hyperbolic partial difference equations of neutral type of the form

$$\begin{aligned} \Delta_2 \left[s_n \Delta_2 \left(u_{m,n} + \sum_{k\in K} r_{k,n} u_{m,n-\alpha_k} \right) \right] + p_{m,n} u_{m,n} + \sum_{i\in I} p_{m,n}^{(i)} f_i\left(u_{m,n-\beta_i}\right) \\ = q_n \nabla^2 u_{m-1,n+1} + \sum_{j\in J} q_{j,n} \nabla^2 u_{m-1,n+1-\gamma_j}, \\ \text{for } (m,n) \in \Omega \times Z^+\left(n_0\right), \end{aligned} \tag{17}$$

with RBC(3) and IC(4).

We can similarly give the definitions of solutions and oscillation. We omit it.

We assume in this section that (H_1)–(H_5) hold, and

(H_6) $s_n \in Z^+(n_0) \to \mathbf{R}^+ \setminus \{0\}$ and $\sum_{n=n_0}^{\infty} 1/s_n = \infty$.

THEOREM 4.1. *Let (H$_1$)–(H$_6$) hold. Suppose that for any constant $A > 0$, there exists an i_0 such that*

$$\sum_{n=n_0}^{\infty} p_{i_0,n} f_{i_0}\left[A\left(1-\sum_{k\in K} r_{k,n-\beta_{i_0}}\right)\right] = \infty. \tag{18}$$

Then every solution of IBVP(17), (3), and (4) is oscillatory in $\Omega \times Z^+(n_0)$.

PROOF. Let $\{u_{m,n}\}$ be such a nonoscillatory solution of IBVP(17), (3), and (4) that $u_{m,n} > 0$ for some $n_1 \in Z^+(n_0)$ and $n \in Z^+(n_1)$. Then we have $u_{m,n-\alpha_k}$, $u_{m,n-\beta_i}$, and $u_{m,n-\gamma_j} > 0$ for $n \in Z^+(n_1+\tau) := Z^+(n_2)$, where $i \in I, j \in J$ and $k \in K$.

Summing (17) in both sides of (17) over Ω, we have

$$\begin{aligned}
\Delta_2 &\left[s_n \Delta_2 \left(\sum_{m\in\Omega} u_{m,n} + \sum_{k\in K} r_{k,n} \sum_{m\in\Omega} u_{m,n-\alpha_k}\right)\right] \\
&+ \sum_{m\in\Omega} p_{m,n} u_{m,n} + \sum_{i\in I}\sum_{m\in\Omega} p_{m,n}^{(i)} f_i\left(u_{m,n-\beta_i}\right) \\
&= q_n \sum_{m\in\Omega} \nabla^2 u_{m-1,n+1} + \sum_{j\in J} q_{j,n} \sum_{m\in\Omega} \nabla^2 u_{m-1,n+1-\gamma_j},
\end{aligned}$$

for $(m,n) \in \Omega \times Z^+(n_2)$. As proved in Theorem 3.1, (7)–(10) hold. Therefore, we obtain

$$\Delta\left[s_n \Delta\left(v_n + \sum_{k\in K} r_{k,n} v_{n-\alpha_k}\right)\right] + \sum_{i\in I} p_{i,n} f_i\left(v_{n-\beta_i}\right) \le 0, \qquad \text{for } n \in Z^+(n_2). \tag{19}$$

Let $w_n = v_n + \sum_{k\in K} r_{k,n} v_{n-\alpha_k}$. Then we have

$$w_n > 0 \quad \text{and} \quad w_n \ge v_n, \qquad \text{for } n \in Z^+(n_2). \tag{20}$$

From (H$_2$), (H$_3$), and (19), we obtain

$$\Delta\left(s_n \Delta w_n\right) \le -\sum_{i\in I} p_{i,n} f_i\left(v_{n-\beta_i}\right) \le 0, \qquad \text{for } n \in Z^+(n_2), \tag{21}$$

which means that $\{s_n \Delta w_n\}$ is decreasing. We claim that

$$s_n \Delta w_n \ge 0, \qquad \text{for } n \in Z^+(n_2). \tag{22}$$

Consequently,

$$\Delta w_n \ge 0, \qquad \text{for } n \in Z^+(n_2). \tag{23}$$

As a matter of fact, if it is not true, then there exists an $n_3 \in Z^+(n_2)$ such that

$$s_{n_3} \Delta w_{n_3} < 0 \quad \text{and} \quad s_n \Delta w_n \ge 0, \qquad \text{for } n_2 \le n < n_3.$$

Using (21), we have

$$\Delta w_n \le \frac{1}{s_n} s_{n_3} \Delta w_{n_3}, \qquad \text{for } n \in Z^+(n_3),$$

which follows that

$$w_{n+1} - w_{n_3} \le s_{n_3} \Delta w_{n_3} \sum_{n=n_3}^{n} \frac{1}{s_n}, \qquad \text{for } n \in Z^+(n_3).$$

Then we have $w_n < 0$ as $n \to \infty$, which contradicts (20).

We know from (21) that for some $i_0 \in I$, we have

$$\Delta\left(s_n \Delta w_n\right) + p_{i_0,n} f_{i_0}\left(v_{n-\beta_{i_0}}\right) \leq 0, \qquad \text{for } n \in Z^+\left(n_3\right), \tag{24}$$

which follows

$$\Delta\left(s_n \Delta w_n\right) + p_{i_0,n} f_{i_0}\left(w_{n-\beta_{i_0}} - \sum_{k\in K} r_{k,n-\beta_{i_0}} v_{n-\beta_{i_0}-\alpha_k}\right) \leq 0, \qquad \text{for } n \in Z^+\left(n_3\right). \tag{25}$$

From (20), (23), and (25), we have

$$\Delta\left(s_n \Delta w_n\right) + p_{i_0,n} f_{i_0}\left[w_{n-\beta_{i_0}}\left(1 - \sum_{k\in K} r_{k,n-\beta_{i_0}}\right)\right] \leq 0, \qquad \text{for } n \in Z^+\left(n_3\right). \tag{26}$$

Summing (26) from n_3 to n and using (23), we have

$$s_{n+1}\Delta w_{n+1} - s_{n_3}\Delta w_{n_3} + \sum_{t=n_3}^{n} p_{i_0,t} f_{i_0}\left[w_{n_3-\beta_{i_0}}\left(1 - \sum_{k\in K} r_{k,t-\beta_{i_0}}\right)\right] \leq 0. \tag{27}$$

By (21) and (22), letting in (27) $n \to \infty$, we have

$$\sum_{t=n_3}^{\infty} p_{i_0,t} f_{i_0}\left[w_{n_3-\beta_{i_0}}\left(1 - \sum_{k\in K} r_{k,t-\beta_{i_0}}\right)\right] < \infty.$$

Let $A = w_{n_3-\beta_{i_0}}$. Then we have

$$\sum_{t=n_3}^{\infty} p_{i_0,t} f_{i_0}\left[A\left(1 - \sum_{k\in K} r_{k,t-\beta_{i_0}}\right)\right] < \infty,$$

which contradicts (18). Thus, this completes the proof.

THEOREM 4.2. *Let (H_1)–(H_6) hold. Suppose that there exists a constant C and some $i_0 \in I$ such that $f_{i_0}(u)/u \geq C$ for $u \neq 0$, and there exists a sequence $b_n \in Z^+(n_0) \setminus \{0\}$ such that*

$$\sum_{n=n_0}^{\infty}\left[C b_{n+1} p_{i_0,n}\left(1 - \sum_{k\in K} r_{k,n-\beta_{i_0}}\right) - \frac{s_{n-\beta_{i_0}}\left(\Delta b_n\right)^2}{4 b_n}\right] = \infty. \tag{28}$$

Then every solution of IBVP(17), (3), and (4) is oscillatory in $\Omega \times Z^+(n_0)$.

PROOF. As shown in Theorem 4.1, we can prove that (18)–(23) hold. From the hypotheses in the theorem and (24), we have

$$\Delta\left(s_n \Delta w_n\right) + C p_{i_0,n}\left(w_{n-\beta_{i_0}} - \sum_{k\in K} r_{k,n-\beta_{i_0}} v_{n-\alpha_k-\beta_{i_0}}\right) \leq 0 \qquad \text{for } n \in Z^+\left(n_3\right). \tag{29}$$

From (20), (23), and (29), we obtain

$$\Delta\left(s_n \Delta w_n\right) + C p_{i_0}\left(1 - \sum_{k\in K} r_{k,n-\beta_{i_0}}\right) w_{n-\beta_{i_0}} \leq 0 \qquad \text{for } n \in Z^+\left(n_3\right). \tag{30}$$

Let $z_n = b_n r_n \Delta w_n / w_{n-\beta_{i_0}}$. Then we have from (23), (29), and (30)

$$\begin{aligned}
\Delta z_n &= \frac{1}{w_{n-\beta_{i_0}} w_{n+1-\beta_{i_0}}} \left\{ \left[b_{n+1} \Delta \left(s_n \Delta w_n \right) + s_n \Delta w_n \Delta b_n \right] w_{n-\beta_{i_0}} - s_n b_n \Delta w_n \Delta w_{n-\beta_{i_0}} \right\} \\
&\le \frac{1}{w^2_{n-\beta_{i_0}}} \left\{ \left[b_{n+1} \Delta \left(s_n \Delta w_n \right) + s_n \Delta w_n \Delta b_n \right] - s_n b_n \Delta w_n \Delta w_{n-\beta_{i_0}} \right\} \\
&\le -C b_{n+1} p_{i_0,n} \left(1 - \sum_{k\in K} r_{k,n-\beta_{i_0}} \right) + \frac{s_n}{w_{n-\beta_{i_0}}} \Delta w_n \Delta b_n - \frac{s_n b_n}{w^2_{n-\beta_{i_0}}} \Delta w_n \Delta w_{n-\beta_{i_0}}.
\end{aligned} \tag{31}$$

By (21), we have

$$\frac{s_n b_n \Delta w_n \Delta w_{n-\beta_{i_0}}}{w_{n-\beta_{i_0}}} \ge \frac{b_n s_n^2 \left(\Delta w_n\right)^2}{s_{n-\beta_{i_0}} w^2_{n-\beta_{i_0}}} = \left(\frac{s_n \Delta w_n}{w_{n-\beta_{i_0}}} \sqrt{\frac{b_n}{s_{n-\beta_{i_0}}}} \right)^2.$$

Also, we have

$$\frac{s_n \Delta w_n \Delta b_n}{w_{n-\beta_{i_0}}} = \sqrt{\frac{b_n}{s_{n-\beta_{i_0}}}} \sqrt{\frac{s_{n-\beta_{i_0}}}{b_n}} \Delta b_n.$$

It follows from the above and (31),

$$\begin{aligned}
\Delta z_n &\le -C b_{n+1} p_{i_0,n} \left(1 - \sum_{k\in K} r_{k,n-\beta_{i_0}} \right) + \frac{s_{n-\beta_{i_0}} \left(\Delta b_n\right)^2}{4 b_n} \\
&\quad - \left(\sqrt{\frac{b_n}{s_{n-\beta_{i_0}}}} \frac{s_n \Delta w_n}{w_{n-\beta_{i_0}}} - \frac{1}{2} \sqrt{\frac{s_{n-\beta_{i_0}}}{b_n}} \Delta b_n \right)^2 \\
&\le - \left[C b_{n+1} p_{i_0,n} \left(1 - \sum_{k\in K} r_{k,n-\beta_{i_0}} \right) - \frac{s_{n-\beta_{i_0}} \left(\Delta b_n\right)^2}{4 b_n} \right].
\end{aligned} \tag{32}$$

Summing (32) from $n_3 + \beta_{i_0}$ to n, we obtain

$$z_{n+1} \le z_{n_3+\beta_{i_0}} - \sum_{t=n_3+\beta_{i_0}}^{n} \left[C b_{t+1} p_{i_0,t} \left(1 - \sum_{k\in K} r_{k,t-\beta_{i_0}} \right) - \frac{s_{t-\beta_{i_0}} \left(\Delta b_t\right)^2}{4 b_t} \right]. \tag{33}$$

Letting in (33) $n \to \infty$ and using (28), we have that $z_n < 0$, which contradicts the definition of z_n. The proof is thus complete.

COROLLARY 4.1. *Let (H_1)–(H_6) hold. If the difference inequality of neutral type (19) has no ultimately positive solution, then every solution of IBVP(17), (3), and (4) is oscillatory in $\Omega \times Z^+(n_0)$.*

5. EXAMPLES

EXAMPLE 5.1. Consider the parabolic equation

$$\begin{aligned}
\Delta_2 \left(u_{m,n} - \frac{1}{2} u_{m,n-1} \right) + u_{m,n} + 2e^{-m^2} u_{m,n-i} e^{u^2_{m,n-i}} &= q_n \nabla^2 u_{m-1,n+1} \\
+ \sum_{j\in J} q_{j,n} \nabla^2 u_{m-1,n+1-\gamma_j}, \qquad \text{for } m = 1, \dots, M \quad &\text{and} \quad n \in Z^+(n_0),
\end{aligned} \tag{34}$$

where $i > [(1/2)e^{M^2} - 1]$ ($[\cdot]$ is the integer function) is an even integer, q_n, $q_{j,n}$, and γ_j satisfy the hypotheses in Theorem 3.1.

Since $r_n = (1/2) < 1$, $p_{m,n} = 1 > B$, $p^*_{m,n} = 2e^{-m^2} \geq B$, where $B := \min\{1, 2e^{-M^2}\}$, $f(u) = ue^{u^2}$, $f(u)/u = e^{u^2} \geq 1 =: C$ and

$$\sum_{s=n-i}^{n} p_{*,s} = (i+1)\, 2e^{-M^2} > \frac{1}{C} = 1,$$

we have that every solution of equation (34) is oscillatory by Theorem 3.1. In fact, $u_{m,n} = (-1)^n m$ is an oscillatory solution of equation (34).

EXAMPLE 5.2. Consider the parabolic equation

$$\Delta_2 \left(u_{m,n} - \frac{n-1}{n} u_{m,n-1} \right) + \frac{1}{2} u_{m,n} + \frac{n^2-2n-2}{2n(n-1)} u_{m,n-1} = q_n \nabla^2 u_{m-1,n+1} + \sum_{j\in J} q_{j,n} \nabla^2 u_{m-1,n+1-\gamma_j}, \qquad \text{for } m = 1,\dots,M \text{ and } \in Z^+(n_0). \tag{35}$$

Note that $r_n = (n-1)/n < 1, f(u) = u$ (consequently, $C = 1$), $p_{m,n} = 1/2 > 1/4, p^*_{m,n} = (n^2-2n-2)/2n(n-1) > 1/4$ for sufficiently large n. However,

$$\lim_{n\to\infty} \left[\frac{n^2-4n+1}{2(n-1)(n-2)} + \frac{n^2-2n-2}{2n(n-1)} \right] = 1.$$

That is, (5) is false at this time. In fact, equation (35) has a nonoscillatory solution $u_{m,n} = mn$.

EXAMPLE 5.3. Consider the hyperbolic equation

$$\Delta_2 \left[n\Delta_2 \left(u_{m,n} + \frac{1}{2} u_{m,n-1} \right) \right] + 3u_{m,n} + \frac{5n^3-18n^2+10n+12}{2(n-1)n(n+1)(n+2)} u_{m,n-2} = q_n \nabla^2 u_{m-1,n+1} + \sum_{j\in J} q_{j,n} \nabla^2 u_{m-1,n+1-\gamma_j}, \tag{36}$$

for $m = 1,\dots,M$ and $n \in Z^+(n_0)$.

It is easy to see that the conditions in Theorem 4.1 are all satisfied. Then every solution of equation (36) is oscillatory. In fact, $u_{m,n} = (-1)^n m/n$ is an oscillatory solution of equation (36).

EXAMPLE 5.4. Consider the hyperbolic equation

$$\Delta_2 \left[n\Delta_2 \left(u_{m,n} + \frac{n-1}{n} u_{m,n-1} \right) \right] + \frac{1}{2} u_{m,n} + m^{1/3}(n-1)^{4/3} \times \frac{n^5 - 11n^4 - 23n^3 - 9n^2 + 8n + 4}{2(n-1)^2 n^2 (n+1)^2 (n+2)^2} u^{2/3}_{m,n-1} = q_n \nabla^2 u_{m-1,n+1} + \sum_{j\in J} q_{j,n} \nabla^2 u_{m-1,n+1-\gamma_j}, \tag{37}$$

for $m = 1,\dots,M$ and $n \in Z^+(n_0)$.

It is easy for one to see that (18) is false in this time. In fact, equation (37) has a nonoscillatory solution $u_{m,n} = m/n^2$.

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